B.Sc./3rd Sem (H)/MATH/22(CBCS)

2022

3rd Semester Examination

MATHEMATICS (Honours)

Paper : C 5-T

(Theory of Real Functions and Introduction to Metric Space)

[CBCS]

Full Marks : 60 — Time : Three Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

Group - A

1. Answer any *ten* questions : 2×10=20

(a) Prove that, $\frac{\tan x}{x} > \frac{x}{\sin x}, 0 < x < \frac{\pi}{2}$

(b) Use Mean value theorem to show that $0 < \frac{1}{x}\log\left(\frac{e^x - 1}{x}\right) < 1$

(c) Give the geometrical interpretation of Mean Value Theorem.

(d) Does there exist a function φ such that $\varphi'(x) = f(x)$ on $[0, 2]$, when $f(x) = x - [x]$, where $[x]$ is the greatest integer function?

(e) Let $I \in \mathbb{R}$ be an interval and a function $f: I \to \mathbb{R}$ be differentiable at $C \in I$. Then if $f'(c) > 0$, prove that f is increasing at c.

(f) Evaluate $\lim_{x \to 3}\left([x] - \left[\frac{x}{3}\right]\right)$

(g) Let $f(x) = \begin{cases} 1, & x \in Q \\ 0, & x \in \mathbb{R} - Q \end{cases}$

Prove that f is discontinuous at every point c in $\mathbb{R}$.

(h) Define the Lipschitz's function.

(i) Find the value of limit : $\lim_{x \to \alpha}\left(1 + \frac{3}{x}\right)^x$

(j) Let d_1 and d_2 are two metrics on a non-empty set A. Prove that $d_1 + d_2$ is also a metric on A.

(k) Let (M, d) be a metric space. Then prove that $\forall A, B \in M, A \subset B \Rightarrow \delta(A) \geq \delta(B)$.

(l) Let (A, d) be a metric space. Then prove that $(A, \sqrt{d})$ is also a metric space.

(m) Prove that in any discrete metric space all the sets are closed.

(n) Let X be a non-empty set and $f: X \to \mathbb{R}$ be an injective function. Then prove that $d(x, y) = |f(x) - f(y)|\ \forall x, y \in X$ defines a metric on X.

(o) Show that $\lim_{x \to \infty} \frac{[x]}{x} = 1$, where $[x]$ denotes the greatest integer contained in x not greater than x.

Group - B

2. Answer any ***four*** questions : 5×4=20

(a) Let $[a, b]$ be a bounded closed Interval and I denoted the set of all Riemann-Integrable $(\mathbb{R}I)$ function over $[a, b]$. Then prove that

$$d(f, g) = \int_a^b |f(x) - g(x)|\,dx, \forall f, g \in \mathbb{R}I$$

is a pseudo metric but not metric.

(b) Prove that $f(x) = \sin x^2$ is not uniformly continuous on $[0, \infty)$ but $f(x) = \sin x$ is uniformly continuous on $[0, \infty)$. 5

(c) State & prove the Hausdorff property.

(d) Let $D \subset \mathbb{R}$ and $f: D \to \mathbb{R}$ be a function. Let c be a limit point of D and $l \in \mathbb{R}$. Then $\lim_{x \to c} f(x) = l$ iff for every sequence $\{x_n\}$ in $D - \{c\}$ converging to c, the sequence $\{f(x_n)\}$ converges to l.

(e) State and prove the Caratheodory's theorem.

(f) Let $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x)=\begin{cases} x^2\sin\frac{1}{x^2}, & x\neq 0 \\ 0, & x=0 \end{cases}$$

Show that f is differentiable on $\mathbb{R}$ but f' is not continuous on $\mathbb{R}$.

Group - C

3. Answer any *two* questions : 10×2=20

(a) (i) State Maclaurin's infinite series and obtain the expansion of $(1+x)^m$ where m is any real number other than positive integer and $|x|<1$.

(ii) Let $f:I\to\mathbb{R}$ be such that f has a local extremum at an interior point C of I. If $f'(c)$ exists then $f'(c)=0$. 5+5

(b) (i) Let I be an interval and a function $f:I\to\mathbb{R}$ be uniformly continuous on I. Then f is continuous on I.

(ii) Let $I=[a, b]$ be a closed and bounded interval and a function $f:I\to\mathbb{R}$ be continuous on I. Then f is uniformly continuous and f is bounded on I. 2+8

(c) (i) Let $I=(a, b)$ be a bounded open interval and $c\in(a,b)$. If $f:I\to\mathbb{R}$ be a monotonic function on I then $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ both exist.

(ii) A function $f:\mathbf{R}\to\mathbf{R}$ is defined by

$$f(x)=\begin{cases} x, & x\in Q \\ 0, & x\in \mathbf{R}-Q \end{cases}$$

Show that f is continuous at 0 and f has a discontinuity of the 2nd kind at every other point in $\mathbb{R}$. 5+5

(d) (i) State and prove Taylor's theorem with Cauchy's form of remainder after n terms.

(ii) Give exampls of a function which is :

(a) Continuous and bounded on $\mathbb{R}$, attains its supremum but not infimum.

(b) Continuous and bounded on $\mathbb{R}$, attains its infimum but not its supremum.
