3. Answer any *one* question : 10×1=10

(a) A projectile of mass m is fired into the air from a gun that is inclined at an angle θ with the horizontal and suppose the initial velocity of the projectile is v_0 feet per second. Neglect all forces except the gravity and the air resistance and assume that this latter force (in pounds) is numerically equal to k times of the velocity (in ft/sec).

(i) Taking the origin at the position of the gun, with the x axis horizontal and the y axis vertical, show that the differential equations of the resulting motion are

$$m\frac{d^2x}{dt^2}+k\frac{dx}{dt}=0,$$

$$m\frac{d^2y}{dt^2}+k\frac{dy}{dt}+mg=0.$$

(ii) Solve the above system of differential equations.

(b) Consider the linear autonomous system

$$\frac{dx}{dt}=x-4y$$

$$\frac{dy}{dt}=2x-5y$$

(i) Determine the nature of the critical point (0, 0) of this system.

(ii) Find the general solution of this system.

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B.Sc./5th Sem (G)/MATH/22(CBCS)

2022

5th Semester Examination

MATHEMATICS (General)

Paper : SEC 3-T

[CBCS]

Marks : 40 Time : Two Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

[Number Theory]

Answer any *five* of the following : 2×5=10

(a) If p is prime and a is prime to p, prove that $a^{p^2-p}\equiv 1\left(\bmod p^2\right)$.

(b) Show that the cube of any integer is of the form $7k$ or $7k\pm 1$.

(c) Show that if m is an integer then 3 divides m^3-m.

(d) Let $\gcd(a, b)=1$. Then show that $\gcd(a+b, a^2+b^2)=1$ *or* 2.

(e) Show that the functions τ and σ are both multiplicative functions.

(f) Show that the number of prime is infinite.

P.T.O.

(g) Show that $89 \mid 2^{44} - 1$.

(h) Find the remainder when 45! is divided by 47?

2. Answer any ***four*** of the following : 5×4=20

(a) Find the solutions of the system of congruences :

$$3x+4y \equiv 5 \pmod{13};\ 2x+5y \equiv 7 \pmod{13}.$$

(b) What is the remainder if $9^{2n+1} + 8^{n+2}$ is divided by 73?

(c) If p is a prime then show that $(p+1)!+1 \equiv 0 \pmod{p}$.

(d) For each positive integer $n \geq 1$, show that $\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } n>1 \end{cases}$ where d runs through the positive divisors of n and μ is the Möbius μ function.

(e) Let $n, p > 1$ are positive integers and p is prime. Given that $n \mid p-1$ and $p \mid n^3 - 1$, prove that $4p-3$ is a perfect square.

(f) If $n \geq 1$ and $g.c.d.(a, n) = 1$ then show that $a^{\phi(n)} \equiv 1 \pmod{n}$. Deduce the Fermat's theorem from it.

3. Answer any ***one*** of the following : 10×1=10

(a) (i) Show that the Goldbach conjecture implies that for each even integer $2n$ there exist integers n_1 and n_2 with $\sigma(n_1)+\sigma(n_2)=2n$. 3

(ii) Find integers x, y such that $5x + 97y = 1$. 3

(iii) Let n be a positive integer. Calculate $gcd\,(n! + 1, (n + 1)!)$. 4

(b) (i) Let $a_n = 6^n + 8^n$. Determine the remainder on dividing a_{83} by 49. 4

(ii) For any positive integer n, show that

$$\frac{\sigma(n!)}{n!} \geq 1+\frac{1}{2}+\frac{1}{3}+\cdots\cdots+\frac{1}{n}.$$ 3

(iii) If 7 does not divides a then show that either $a^3 + 1$ or $a^3 - 1$ is divisible by 7. 3

OR

[Bio-Mathematics]

1. Answer any *five* questions : 2×5=10

(a) The characteristic equation of a system is given as follows $x^3+3x^2+7x+k=0$. Using the Routh-Hurwitz Stability Criteria, find the range of values of k for which the system would be stable.

(b) Let f be differentiable. Prove that a stationary point $\bar{x}$ of $x_{n+1}=f(x_n)$ is locally asymptotically stable, if $|f'(\bar{x})|<1$ and unstable, if $|f'(\bar{x})|>1$.

(c) What is Holling type II functional response?

(d) Define asymptotically stable steady state solution of a dynamical system.

(e) Show that for a decaying population

$$\frac{dN}{dt}=-KN \qquad (K>0)$$

the time at which only half of the original population remains (the half-life) is $T_{1/2}=\frac{\ln 2}{K}$.

(f) Find the steady states of the following system of equations and determine the Jacobian of the system for these steady states :

$$\frac{dx}{dt}=x-xy,$$

$$\frac{dy}{dt}=xy-y.$$

(g) Define basic reproduction number and write down the significance of it.

(h) Write the basic differences between Malthus model and logistic growth model.

2. Answer any *four* questions : 5×4=20

(a) What is intra species competition? Give an example of two species competition model. 3+2

(b) Sketch the phase-plane behaviour of the following system of the linear equations and classify the stability characteristic of the steady state at (0, 0) :

$$\frac{dx}{dt}=5x+8y,$$

$$\frac{dy}{dt}=-3x-5y.$$ 5

(c) Write a short note on Allee effect. 5

(d) What is diffusion in mathematical model? Give an example of one species model with diffusion. 3+2

(e) A delayed two species model is given by :

$$\frac{dX}{dt}=X(t)\left[1-\sigma X(t)-\lambda Y(t)\right];\ \sigma>0,\ \lambda>0$$

$$\frac{dY}{dt} = \alpha X(t-\tau)Y(t-\tau) - \mu Y(t);\ \alpha, \beta > 0$$

Find the steady states and analyze the stability of non-zero steady state. 5

(f) Deduce Fisher's equation for spreading genes. 5

3. Answer any *one* question : 10×1=10

(a) What do you mean by SIR model in epidemiology? Write down the basic equations of SIR model and find an expression of number of infectives at a certain time. 3+7

(b) (i) Find out the steady state solutions and discuss the stability of the following prey-predator model :

$$\frac{dX}{dt} = rX(a - bY),$$

$$\frac{dY}{dt} = -cY + dXY.$$

where X and Y are the density of prey and predator respectively and all the parameters a, b, c, d and r are positive.

(ii) Give the geometrical interpretation of the above prey-predator model. 6+4

OR

[Mathematical Modelling]

1. Answer any *five* questions : 2×5=10

(a) What is Damped Forced Oscillation?

(b) What is Balance Law in mixing problem?

(c) Show that the two solutions $y = \sin 3t$ and $y = \cos 3t$ of $y'' + 9y = 0$ are linearly independent for all values of t.

(d) If the partial differential equation $\frac{\partial \rho}{\partial t} + \frac{\partial(\rho V)}{\partial x} = 0$, $\rho(x, 0) = f(x)$ denotes a Traffic flow problem, then define all the variables.

(e) Write Euler equations for system of conservation laws : gas dynamics.

(f) Write Kirchhoff's Voltage law.

(g) Write one dimensional heat equation for partial differential equation (PDE). Write nature of this partial differential equation (PDE).

(h) Explain stability of linear simultaneous differential equation.

2. Answer any *four* questions : 5×4=20

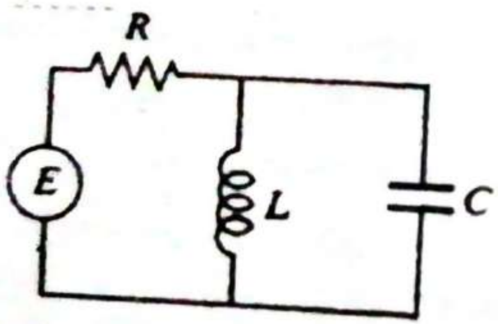
(a) Obtain the solution of the wave equation $u_{tt} = c^2 u_{xx}$, under the following conditions :

(i) $u(0, t) = u(2, t) = 0$.

(ii) $u(x, 0) = \sin^3 \frac{\pi x}{2}$.

(iii) $u_t(x, 0) = 0$.

(b) For the given network below assume, E is an electromotive force of 15V, R is a resistor of 20Ω, L is an inductor of 0.02 H and C is a capacitor of 10^{-4} farads. Set up the differential equation for the currents.



(c) A tank of water is 100 gallons. Each minute, 1 gallon of water enters the tank with 5 grams of pollutants. The tank of water is evenly mixed. Each minute, one gallon of the even mixture drains from the tank. If the tank starts full of water without pollutants, how many grams of pollutants are there in the tank at time t ?

(d) A 16-lb weight is attached to the lower end of a coil spring suspended from the ceiling, the spring constant of the spring being 10 lb/ft. The weight comes to rest in its equilibrium position. Beginning at $t = 0$ an external force given by $F(t) = 5\cos 2t$ is applied to the system. Determine the resulting motion if the damping force in pounds is numerically equal to $2\frac{dx}{dt}$. Where $\frac{dx}{dt}$ is the instantaneous velocity in feet per second.

(e) Solve the equation $\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = 10\sin 2t$.

The initial conditions are $x(0) = \frac{1}{2}$, $x^1(0) = 0$, where $x^1(t) = \frac{dx}{dt}$.

(f) (i) Eliminate the arbitrary constants of $z = (x^2 + a)(y^2 + b)$ and hence obtain the corresponding PDE.

(ii) Eliminate the arbitrary function of $z = x + y + f(xy)$ and hence obtain the corresponding PDE.