(ii) If α, β be any two roots of the equation $x^3 + qx + r = 0$. Find the equation whose roots are the six values of $\frac{\alpha}{\beta}$. 5+5

(c) (i) Show that the roots of the cubic $x^3 - 3a^2x - 2a^3 \cos 3A = 0$ are $2a \cos A$, $2a\cos(120° \pm A)$.

(ii) If α be an imaginary root of the equation $x^5 - 1 = 0$, find the equation whose roots are $\alpha + 2\alpha^4$, $\alpha^2 + 2\alpha^3$, $\alpha^3 + 2\alpha^2$, $\alpha^4 + 2\alpha$. 5+5

(d) (i) Show that the roots of the equation $(x-a)(x-b)(x-c) - f^2(x-a) - g^2(x-b) - h^2(x-c) + 2fgh$ are all real.

(ii) Prove that if two roots of Euler's cubic vanish, then the biquadratic $a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$ has two pairs of equal roots given by $\frac{-a_1 \pm \sqrt{(-3H)}}{a_0}$ 5+5

B.Sc./5th Sem (H)/MATH/22(CBCS)

2022

5th Semester Examination

MATHEMATICS (Honours)

Paper : DSE 1-T

[CBCS]

Full Marks : 60 Time : Three Hours

The figures in the margin indicate full marks. Candidates are required to give their answers in their own words as far as practicable.

[Linear Programming]

1. Answer any ***ten*** questions : 2×10=20

(a) Define convex set.

(b) What is an extreme point in E^n?

(c) What is the dual of an LPP?

(d) Define the saddle point of a matrix game.

(e) Solve the LPP by graphical method :

$$\text{Max } Z = 5x_1 + 3x_2$$
$$\text{Subject to } 3x_1 + 5x_2 \le 15$$
$$5x_1 + 2x_2 \le 10$$
$$x_1, x_2 \ge 0$$

(f) When does a set of vectors form a basis of E^n?

(g) Prove that a hyperplane is a convex set.

(h) Is assignment problem a Linear Programming problem? Justify.

(i) Show that the convex hull of two points x_1 and x_2 is the line segment joining these points.

(j) Explain what is meant by a transportation problem.

(k) Is the solution $x_1 = -6,\ x_2 = 0,\ x_3 = 4$ a basic solution of the following equations?

$$x_1 + 2x_2 + 3x_3 = 6$$
$$2x_1 + x_2 + 4x_3 = 4$$

(l) State the fundamental theorem of LPP.

(m) When a LPP is said to be has an unbounded solution?

(n) Show that the LPP Max $z = 3x_1 + 9x_2$

subject to $x_1 + 4x_2 \le 8$

$x_1 + 2x_2 \le 4$

$x_1,\ x_2 \ge 0$

admits of a degenerate basic feasible solution.

(o) Write down the transportation problem

	D_1	D_2	
O_1	3	1	10
O_2	2	4	8
	12	6	

into LPP.

2. Answer any ***four*** questions : 5×4=20

(a) Given a basic feasible solution $X_B = B^{-1}b$ with $Z_0 = C_B X_B$ to the LPP Max $Z = CX$ subject to $AX = b,\ X \ge 0$ and $z_j - c_j \ge 0$ for every column a_j in A. Prove that z_0 is the maximum value of Z.

(b) Show that by the simplex method, the following LPP admits more than one optimum solution.

Max $Z = 2x_1 + 3x_2$

$x_1 + 3x_2 \le 21$

$2x_1 + 3x_2 \le 24$

$x_1 + x_2 \le 10$

$5x_1 + 4x_2 \le 48$

$x_1 \ge 0,\ x_2 \ge 0$

(c) Prove that if the primal problem has an unbounded objective function then the dual has no feasible solution.

(d) Using two phase method, show that feasible solution does not exist to the problem

Min $Z = x_1 + x_2$

subject to $3x_1 + 2x_2 \ge 30$

$2x_1 + 3x_2 \ge 30$

$x_1 + x_2 \le 5,\ x_1 \ge 0,\ x_2 \ge 0.$

(e) Formulate the dual of the following LPP and hence solve it.

Maximize $Z = 3x_1 - 2x_2$

subject to $x_1 \le 4$

$x_2 \le 6$

$x_1 + x_2 \le 5$

$-x_2 \le -1$

$x_1, x_2 \ge 0$

(f) Solve the following game graphically

B

A	1	3	0	2
	3	0	1	–1

3. Answer any *two* questions : 10×2=20

(a) (i) Use Charne's Big-M method to

Minimize $Z = 2x_1 + 4x_2 + x_3$

subject to $x_1 + 2x_2 - x_3 \le 5$

$2x_1 - x_2 + 2x_3 = 2$

$-x_1 + 2x_2 + 2x_3 \ge 1$

$x_1, x_2, x_3 \ge 0$ 8

(ii) Determine the position of the point (–6, 1, 7, 2) relative to the hyperplane

$3x_1 + 2x_2 + 4x_3 + 6x_4 = 7$ 2

(b) Solve the following transportation problem

	D_1	D_2	D_3	D_4	a_i
O_1	3	8	7	4	30
O_2	5	2	9	5	50
O_3	4	3	6	2	80
b_j	20	60	55	40	

Is there any alternative optimal solution to the problem? 8+2

(c) (i) Find the assignments to find the minimum cost for the assignment problem with the following cost matrix.

	A	B	C	D	E
1	6	5	8	11	16
2	1	13	16	1	10
3	16	11	8	8	8
4	9	14	12	10	16
5	10	13	11	8	16

7

(ii) Write a short note on degeneracy in LPP. 3

(d) (i) Prove that if a fixed number P is added to each element of the pay-off matrix then the value of the game is increased by P while the optimal strategies remains unchanged. 5

(ii) Use dominance to reduce the pay-off matrix

A

B	–5	3	1	20
	5	5	4	6
	–4	–2	0	–5

and hence solve. 5

OR

[Point Set Topology]

Group - A

1. Answer any ***ten*** questions : 2×10=20

(a) Is union of two topologies on X, a topology? Answer with justification.

(b) Show that each function on [0, 1] onto $\{1, \frac{1}{2}, \frac{1}{3}, ...\}$ must fail to be continuous.

(c) Give definition of a maximal element and least upper bound of a partial ordered set. Give an example of a set with least upper bound but having no maximal element.

(d) Define the subspace topology with an example.

(e) If α, β and γ be the cardinal numbers, prove that $\left(\alpha^{\beta}\right)^{\gamma} = \alpha^{\beta\gamma}$.

(f) If $X = \{a, b, c\}$, $\tau = \{\varphi, X, \{a\}, \{b, c\}\}$, then find the derived sets of all subsets of X.

(g) Show that the union of two connected subsets of a space need not be connected.

(h) Find the interior of the set [0, 1] in the lower limit topology on $\mathbb{R}$.

(i) List five distinct non-trivial topologies for the set $X = \{a, b, c, d\}$.

(j) Show that in a topological space X, $X \setminus \bar{A} = (X \setminus A)^{\circ}$, $A \subset X$.

(k) Show that every indiscrete topological space is compact.

(l) If $X = \{a, b, c\}$, $\tau = \{\varphi, X, \{a\}, \{b,c\}\}$, then show that the topological space (X, τ) is disconnected.

(m) State the Lebesgue number lemma.

(n) If $\tau = \{\varphi, X, \{1\}, \{2,3\}, \{1,2,3\}, \{2,3,4,5\}\}$ is a topology on $X = \{1, 2, 3, 4, 5\}$, then find the components of X.

(o) Show that the map $f:(\mathbb{R},U) \to (\mathbb{R},U)$ given by

$$f(x) = x \text{ if } x < 1$$

$$= 1 \text{ if } 1 \le x \le 2$$

$$= \tfrac{x^2}{4} \text{ if } x > 2$$

is continuous but not open, here, U denotes the usual topology.

Group - B

2. Answer any ***four*** questions : 5×4=20

(a) If $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{R}$ denotes the set of real numbers then prove that

(i) cardinality of $2^{\mathbb{N}}$ = cardinality of $\mathbb{R}$.

(ii) cardinality of $\mathbb{R}^{\mathbb{N}}$ = cardinality of $\mathbb{R}$. 3+2

(b) Let X be a non-empty set and a mapping $C: P(X) \to P(X)$ satisfying

(i) $A \subset C(A) \forall A \in P(X)$

(ii) $C(A) = \varphi$

(iii) $C(A \cup B) = C(A) \cup C(B) \forall A, B \in P(X)$

(iv) $C(C(A)) = C(A) \forall A \in P(X)$

Then show that there exists a unique topology τ on X such that for each $A \subset X$, $C(A) = \bar{A}^{\tau}$. 5

(c) Prove that a family β of subsets of X containing φ is a base for some topology on X if and only if

(i) For any two elements $B_1, B_2 \in \beta$ and for any $x \in B_1 \cap B_2$, $\exists B_3 \in \beta$ such that $x \in B_3 \subset B_1 \cap B_2$.

(ii) $X = \bigcup\{B : B \in \beta\}$. 5

(d) Let (X, τ) be a product topological space of the family of spaces $\{\langle X_i, \tau_i\rangle : i \in I\}$. Then prove that

(i) $\Pi_i : \langle X, \tau\rangle \to \langle X_i, \tau_i\rangle$ is continuous for each $i \in I$.

(ii) τ is the weakest topology on X, such that each $\Pi_i : \langle X, \tau\rangle \to \langle X_i, \tau_i\rangle$ is continuous.

(iii) $\Pi_i : \langle X, \tau\rangle \to \langle X_i, \tau_i\rangle$ is an open mapping.

2+2+1

(e) Prove that every real-valued continuous function over a compact space is bounded and attains its bounds. Give an example to show that the result does not hold if the compactness of the space is withdrawn. 5

(f) Let (X, d) be a metric space and $A \subseteq X$. Prove that if A is totally bounded then A is bounded. Is the converse true? Justify with an example. 2+3

3. Answer any ***two*** questions : 10×2=20

(a) (i) Prove that Zorn's Lemma (Restricted form) ⇒ Hausdorff maximality principal ⇒ Zorn's Lemma (General form).

(ii) State and prove Schoder-Bernstine theorem.

5+5

(b) (i) Prove that a function $f : \langle X, \tau_1\rangle \to \langle X, \tau_2\rangle$ is continuous if and only if for any $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$. Hence show that a bijection function f is a homeomorphism if and only if $f(\overline{A}) = \overline{f(A)}$.

(ii) Prove that every closed surjective continuous function is a quotient map. Is the converse true? Answer with justification. (3+2)+(3+2)

(c) (i) Prove that continuous image of a connected set is connected. Hence show that a continuous real valued function defined over a connected set possesses intermediate value property.

(ii) Prove that a set in the space of real with usual topology is connected if and only if it is an interval. (3+2)+5

(d) Let (X, d) be a metric space and $A \subseteq X$. Then prove that

(i) If A is compact, then A is totally bounded.

(ii) If X is complete and A is totally bounded, then A is compact in X. 5+5

———

OR

[Theory of Equations]

1. Answer any ***ten*** questions from the following : 2×10=20

(a) If $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equation $x^n + nax + b = 0$ prove that $(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3) \ldots (\alpha_1 - \alpha_n) = n(\alpha_1^{n-1} + a)$.

(b) Apply Descarte's rule of signs to find the nature of the roots of the equation $x^6 + 7x^4 + x^2 + 2x + 1 = 0$.

(c) Solve the equation $x^4 + x^2 - 2x + 6 = 0$, it is given that $1 + i$ is a root.

(d) How many times the graph of the polynomical $(x^3 - 1)(x^2 + x + 1)$ will cross x-axis.

(e) Let $f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$, where $a_0, a_1, \ldots, a_n$ are integers. If $\frac{p}{q}$ be a rational root of the equation $f(x) = 0$, where p and q are prime to each other, then prove that p is a divisor of a_n.

(f) Find the equation whose roots are the squares of the roots of the cubic $x^3 - ax^2 + bx - 1 = 0$.

(g) The roots of the equation $x^3 - 3px^2 + 3(p-1)x + 1 = 0$ are α, β, γ, find the equation whose roots are $1-\alpha, 1-\beta, 1-\gamma$.

(h) If p, q, r be positive, then find the nature of the roots of the equation $x^4 + px^3 + qx - r = 0$.

(i) If $p_r = p_{r+1} = p_{r+2}$, prove that $x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_n = 0$ cannot have more than $n-2$ real roots.

(j) The equation $x^n - nx + n - 1 = 0$, $(n > 1)$ is satisfied by $x = 1$. What is the multiplicity of this root?

(k) If α be an imaginary root of $x^{11} - 1 = 0$, prove that $(\alpha + 1)(\alpha^2 + 1) \cdots (\alpha^{10} + 1) = 1$.

(l) If α, β, γ are the roots of $x^3 + 3x + 2 = 0$, then find the value of $\sum \alpha\beta(\alpha + \beta)^3$.

(m) If α, β, γ are the roots of $x^3 - ax^2 + bx - c = 0$, then prove that area of the triangle whose sides are α, β, γ is $\frac{1}{4}\left[a(4ab - a^3 - 8c)\right]^{1/2}$.

(n) Find an upper limit of the real roots of the equation $x^4 - 2x^3 + 3x^2 - 2x + 2 = 0$.

(o) If $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the roots of the equation $x^n + \frac{x^{n-1}}{1!} + \frac{x^{n-2}}{2!} + \cdots + \frac{1}{n!} = 0$ and $S_r = \sum \alpha_1^r$, show that $S_r = 0$ for $r = 2, 3, \cdots, n$ but $S_r \neq 0$ for $r = n+1, n+2, \cdots$.

2. Answer any ***four*** questions : 5×4=20

(a) If α, β, γ be the roots of the equation $x^3 - 3x + 1 = 0$, then prove that

$$\begin{vmatrix} \alpha^3 & \alpha^2 & 1 \\ \beta^3 & \beta^2 & 1 \\ \gamma^3 & \gamma^2 & 1 \end{vmatrix} = \pm 27.$$

(b) Obtain the equation whose roots are the square of the roots of the equation $x^4 - x^3 + 2x^2 - x + 1 = 0$. Use Descarte's rule of signs to the resulting equation to deduce that the given equation has no real root.

(c) If $\alpha, \beta, \gamma, \delta$ be the roots of $x^4 + px^3 + qx^2 + rx + s = 0$, then find the equation whose roots are $\beta\gamma + \alpha + \delta$, $\gamma\alpha + \beta\delta$, $\alpha\beta + \gamma\delta$.

(d) If the equation $x^n - p_1 x^{n-1} + p_2 x^{n-2} - p_3 x^{n-3} + \cdots = 0$ has n positive distinct roots, then prove that $p_1, p_2, \cdots$ are all positive and $p_1^2 - 2p_2 > 0$.

(e) Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the roots of the equation $f(x) = x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_n = 0$ and let $s_r = \alpha_1^r + \alpha_2^r + \cdots + \alpha_n^r$ where r is an integer ≥ 0. Then prove that $s_r + p_1 s_{r-1} + p_2 s_{r-2} + \cdots + p_n s_{r-n} = 0$, if $r \geq n$.

(f) If $\alpha + \beta + \gamma + \delta = 0$, prove that $\frac{\alpha^5 + \beta^5 + \gamma^5 + \delta^5}{5} = \frac{\alpha^3 + \beta^3 + \gamma^3 + \delta^3}{3} \cdot \frac{\alpha^2 + \beta^2 + \gamma^2 + \delta^2}{2}$. Also find the value of $\sum \alpha^7$.

3. Answer any ***two*** questions : 10×2=20

(a) (i) Let $f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$, where $a_0, a_1, \ldots, a_n$ are integers. If $f(0)$ and $f(1)$ are both odd, prove that the equation can not have an integer root. Hence prove that the equation $x^4 + 6x^3 + 3x^2 - 14x + 15 = 0$ cannot have an integer root.

(ii) Apply Descarte's rule of signs to find the nature of the roots of the equation $x^8 + 1 = 0$.

(4+3)+3

(b) (i) Find the number and position of the real roots of the equation $x^3 - 3x^2 - 4x + 13 = 0$ by using Sturm's method.