

2022

5th Semester Examination
MATHEMATICS (Honours)

Paper : C 12-T

[Group Theory II]

[CBCS]

Full Marks : 60

Time : Three Hours

*The figures in the margin indicate full marks.
Candidates are required to give their answers
in their own words as far as practicable.*

Group - A

1. Attempt any *ten* questions : 2×10=20
- (a) Find two non-isomorphic groups H_1 and H_2 such that $Aut(H_1)$ is isomorphic with $Aut(H_2)$.
 - (b) Let G be a group. Then prove that $|Inn(G)| = 1$ if and only if G is commutative.
 - (c) Verify whether $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is isomorphic to \mathbb{Z}_4 .
 - (d) Let G be a cyclic group of order 2023. Find the number of automorphisms defined on G .

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- (e) Express $U(165)$ as an external direct product of cyclic groups of the form Z_n .
- (f) Give an example of a group G such that $|G| = 12$ and G has more than one subgroup of order 6.
- (g) Define characteristic subgroup of a group G . Is it true that every normal subgroup is characteristic? Give reasons in support of your answer.
- (h) Find the class equation for the Klein's four group.
- (i) Let p, q be odd primes and let m and n be positive integers. Is $U(p^m) \times U(q^n)$ cyclic? Justify your answer. Here $U(n)$ denotes the group of units modulo n .
- (j) Let G be a p -group (where p is a prime) and H be a non-trivial homomorphic image of G . Then prove that H is also a p -group.
- (k) Let R denote the set of all polynomials with integer coefficients in the independent variables x_1, x_2, x_3 . Let S_3 act on R by $\sigma \cdot p(x_1, x_2, x_3) = p(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$. Find the stabilizer of the polynomial $x_1 x_2$ under the action of G .
- (l) Express the Klein's four group as an internal direct product of two of its proper subgroups.

- (m) Find the conjugacy classes of $cl((1,2))$ and $cl((1,2,3))$ in S_3 .
- (n) Verify whether a non-commutative group of order 343 is simple.
- (o) State fundamental theorem for finite abelian groups.

Group - B

2. Attempt any *four* questions : 5×4=20
- (a) Prove that commutator subgroup G' of a group G is a characteristic subgroup of G . 5
- (b) Let G be a group. Define commutator subgroup of G . Prove that Commutator subgroup G' is a normal subgroup of G and G/G' is commutative. 1+4
- (c) Find all subgroups of order 3 in $Z_9 \oplus Z_3$. 5
- (d) Determine all non-isomorphic abelian groups of order 720. 5
- (e) Let G be a group of order 60. If Sylow 3-subgroup is normal in G then show that Sylow 5-subgroup is also normal in G . 5
- (f) Let G be a group. Prove that the mapping $\phi: G \times G \rightarrow G$ defined by $\phi(g, a) = g \cdot a = gag^{-1}$ is a group action. Find its kernel and stabilizer G_g . 5

Group - C

3. Attempt any *two* questions : 10×2=20

- (a) (i) Find $Aut(\mathbb{Z})$.
- (ii) If G is a non-abelian group then show that $Aut(G)$ can not be cyclic.
- (iii) Prove that $Inn(G) \approx \frac{G}{Z(G)}$, where $Inn(G)$ is the group of inner automorphism of G and $Z(G)$ is the centre of G . 2+3+5
- (b) (i) Show that $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ is not isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_4$.
- (ii) Find all conjugacy classes of the Dihedral group D_8 of order 8 and hence verify the class equation.
- (iii) Let G and H be finite cyclic groups. Prove that $G \oplus H$ is cyclic if and only if $|G|$ and $|H|$ are relatively prime. 2+3+5
- (c) (i) State Cauchy's theorem. Use Cauchy's theorem to prove that if a finite group G is a p -group then $|G| = p^n$ for some positive integer n , where p is a prime.
- (ii) Let G be a group of order pn , where p is a prime and $p > n$. Show that there exists a subgroup of order p in G which is normal.

(iii) Find the number of elements of order 5 in the direct product $\mathbb{Z}_{25} \oplus \mathbb{Z}_5$. 3+3+4

- (d) (i) Let G be a group acting on a non-empty set S and $a \in S$. Then prove that $|[a]| = [G : G_a]$ where $[a]$ denotes the orbit of a and G_a denotes the stabilizer of a .
- (ii) Let G be a finite group and H be a proper subgroup of G with index n such that $|G|$ does not divide $n!$. Using group action show that G contains a non-trivial normal subgroup. Hence show that a simple group of order 63 cannot contain a subgroup of order 21. 4+(3+3)