বিদ্যাসাগর বিশ্ববিদ্যালয়

VIDYASAGAR UNIVERSITY

Question Paper

B.Sc. Honours Examination 2022

(Under CBCS Pattern)

Semester - VI

Subject : MATHEMATICS

Paper : C 13 - T

Full Marks : 60

Time : 3 Hours

Candidates are required to give their answers in their own words as far as practicable.
The figures in the margin indicate full marks.

[METRIC SPACES AND COMPLEX ANALYSIS]

Group - A

1. Answer any ***five*** questions : 2×5=10

(a) Let $A = \{(x, y) : x, y \in \mathbb{R},\ x \notin \mathbb{Q} \ or \ y \notin \mathbb{Q}\}$. Examine whether A is compact or not in $\mathbb{R}^2$ with the usual metric.

(b) Consider $\mathbb{R}^2$ with the usual metric. Examine whether the set

$$A = \left\{\left(x, \sin\frac{1}{x}\right) \in \mathbb{R}^2 : 0 < x < 1\right\}$$ is connected or not.

(c) Define finite intersection property. Does the collection $A = \{(n-1, n+1) : n \in Z\}$ of open in intervals satisfy finite intersection property? Justify.

(d) Show that every Cauchy sequence in a metric space is bounded.

(e) Find the radius of convergence of the power series $\sum_{n=1}^{\infty}(4+3i)^n z^n$.

(f) Let $T(z) = \frac{az+b}{cz+d}$ be a bilinear transformation. Show that ∞ is a fixed point of T if and only if $c = 0$.

(g) Let $f'(z) = 2x + ixy^2$ where $z = x + iy$. Show that $f'(z)$ does not exist at any point of z-plane.

(h) Show that $f(z) = e^{-|z|^4} + z + 5$ is not differentiable at any non-zero point.

2. Answer any ***four*** questions : 5×4=20

(a) (i) Prove that a metrix space (X, d) having the property that every continuous map $f : X \to X$ has a fixed point, is connected. 2

(ii) Let (X, d) be a complete metric space and $T : X \to X$ be a contraction on X. Then for $x \in X$, show that the sequence $\{T^n(x)\}$ is a convergent sequence. 3

(b) Let (X, d_1) and (Y, d_2) be two metric spaces and $f : (X, d_1) \to (Y, d_2)$ be uniformly continuous. Show that if $\{x_n\}$ is a Cauchy sequence in (X, d_1) then so is $\{f(x_n)\}$ in (Y, d_2). Is it true if f is only continuous? Justify. 3+2

(c) Show that continuous image of a compact metric space is compact. 5

(d) Let $f(z) = u + iv$ be analytic in a domain D. Prove that f is constant in D if and only if one of the following holds :

(i) $f'(z)$ vanishes in D.

(ii) *Ref (z) = u = constant.*

(iii) *Imf (z) = v = constant.*

(iv) | *f(z)* | = *constant (non zero)*

Check the analyticity of $f(z)=\overline{z}$. 5

(e) Find the domain of convergence of the following series

$$\sum_{n=1}^{\infty}\frac{1.3.5.....(2n-1)}{n!}\left(\frac{1-z}{z}\right)^n.$$ 5

(f) Evaluate :

(i) $\int_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{(z-1)(z-2)}dz$ where C is the circle $|z|=3$ described in the positive sense.

(ii) $\int_C \frac{zdz}{(9-z^2)(z+i)}$, where C is the circle $|z|=2$ described in the positive sense.

2+3

3. Answer any ***three*** questions : 10×3=30

(a) (i) Let (*X, d*) be a metric space and *A* be a compact subset of *X*. Show that *A* is totally bounded.

(ii) A subset *A* of a metric space (*X, d*) is totally bounded if and only if every sequence in *A* has a Cauchy sequence.

(iii) If $\mathcal{F}$ be a family of compact sets with finite intersection property in a metric space (*X, d*), then show that $\cap\mathcal{F}\neq\phi$. 2+5+3

(b) (i) Show that the map $f:[0,1]\to[0,1]$, defined by $f(x)=x-\frac{x^2}{2}, x\in(0,1)$ is a weak contraction but not a contraction map.

(ii) Let (*X, d*) be a complete metric space and $f:X\to X$ be a contraction map with Lipschitz constant $t(0<t<1)$. If $x_0\in X$ is the unique fixed point of *f*, show that $d(x, x_0) \le \frac{1}{1-t} d(x, f(x))$, for all $x \in X$.

(iii) Show that a contraction of a bounded plane set may have the same diameter as the set itself. 3+5+2

(c) (i) Let $f(z) = u(x, y) + iv(x, y), z = x + iy$ and $z_0 = x_0 + iy_0$. Let the function f be defined in a domain D except possible at the point z_0 in D. Then prove that $\lim_{z \to 0} f(z) = u_0$ if and only if $\lim_{x \to x_0} u(x, y) = u_0$ and $\lim_{y \to y_0} v(x, y) = v_0$.

(ii) If $f(z) = u(x, y) + iv(x, y)$ is an analytic function of $z = x + iy$ and $u(x, y) - v(x, y) = \frac{e^y - \cos x + \sin x}{\cosh y - \cos x}$ find *f(z)* subject to the condition $f\left(\frac{\pi}{2}\right) = \frac{3-i}{2}$. 5+5

(d) (i) Suppose *f(z)* is analytic in a domain Ω and $C = \{z : |z - a| = R\}$ contained in Ω. Then prove that $\left|f^n(a)\right| \le \frac{n! M_R}{R^n}, n = 0, 1, 2, ...$

where $M_R = \max_{z \in C} |f(z)|$.

(ii) Show that every bounded entire function is constant.

(iii) Let $p(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n, a_n \ne 0$. Show that there exists a point z_0 in C such that $p(z_0) = 0$. 3+3+4

(e) (i) Show that when $0 < |z| < 4$, $\frac{1}{4z - z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}$.

(ii) Find the Laurent series that represents the function $f(z) = z^2 \sin\left(\frac{1}{z^2}\right)$ in the domain $0 < |z| < \infty$. 5+5